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Steady inviscid rotational flows with free surfaces

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An inviscid fluid in steady two-dimensional motion in a region bounded by a closed streamline must have constant vorticity. We solve here for some such flows where the boundary is in part free, the fluid velocity magnitude being constant on the free boundary. A trivial example of such a flow is a circular cylinder of fluid rotating about its axis as if rigid, for which the whole circular boundary is free, irrespective of its radius. We now ask what happens to that flow when it comes into contact with solid boundaries. There is no steady flow when the contact is with a finite segment of a single plane wall, but a unique solution exists when the rotating fluid mass is in contact with some concave boundaries. Computed results are obtained for vortices lying inside a parabolic dish, or in a corner between two planes.

1. Introduction

Batchelor (1956) has shown that an inviscid fluid must have constant vorticity in two-dimensional steady flow within a region bounded by closed streamlines. Such Batchelor (sometimes 'Prandtl–Batchelor') flows have since been studied by a number of authors (see Moore, Saffman & Tanveer 1988, and earlier studies referenced in that paper) as models of wakes, where the closed rotational region lies within a region of irrotational flow. For definiteness, we shall refer in this paper to the region of rotational flow as a vortex.

A somewhat simpler case is where the boundary of the vortex is (in principle) not in contact with any other fluid. Then this boundary is either one of contact with a solid (impermeable but slip-permitting) wall, or is a free boundary of constant pressure. For example, it has been conjectured (Tuck 1992) that the foamy area or 'bow wake' (Mori 1985) at the extreme bow of a bluff ship or barge may have this character, as in figure 1. This problem can be solved by decomposing the flow domain into two parts: the irrotational flow below the dividing streamline, and the vortex. The irrotational flow can be computed by adapting the boundary integral equation method described in Tuck & Vanden-Broeck (1984). The vortex is then a flow bounded by a free surface, a rigid surface and a portion of the dividing streamline. The irrotational flow and the vortex are coupled by the dividing streamline whose shape needs to be adjusted to satisfy the continuity of the pressure. However, as a first attempt to model this situation, we may replace the dividing streamline by a fixed boundary, yielding an idealized flow such as that in figure 2.

In the above application, gravity plays a role, but for the present study we neglect the effect of gravity on the vortex. In effect, we are assuming that the lengthscale of the vortex is small compared to the gravitational lengthscale, so that the local Froude number is high. Gravity enters the problem through the constant-pressure free-surface condition (Bernoulli's equation). In the absence of gravity it simplifies to one of constancy of the fluid velocity magnitude.

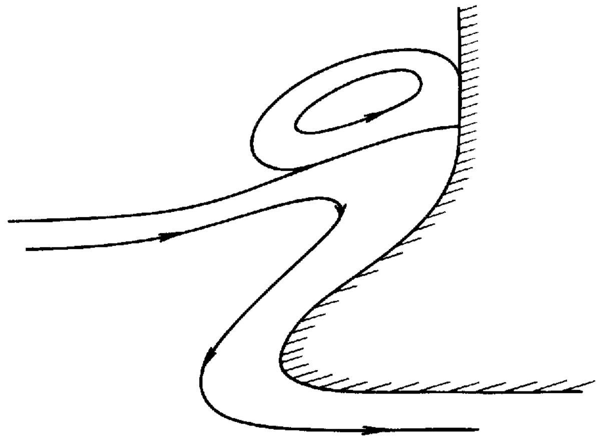

FIGURE 1. Sketch of conjectured ship bow flow.

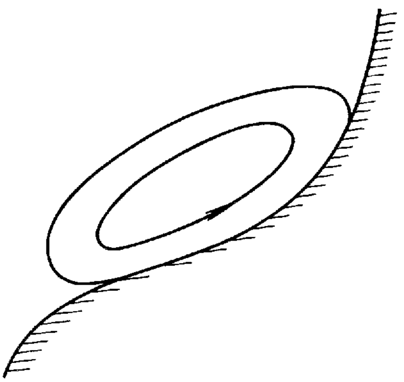

FIGURE 2. Sketch of vortex in contact with a wall of general shape.

A trivial example of a Batchelor flow is a 'free vortex', consisting of a circular cylinder of fluid rotating as if rigid. Then a cylindrical boundary of any radius R is a possible free boundary of constant velocity magnitude. Our concern in the present paper is (in effect) with distortions of this free vortex produced by bringing it in contact with rigid boundaries.

If the (constant) vorticity is ω and the vortex occupies area A, we take $R = (A/\pi)^{\frac{1}{2}}$ as lengthscale and ωR as velocity scale. In this non-dimensional framework, all vortices of interest have unit vorticity and area π. Thus, for example, the free cylindrical vortex is now of unit radius, and has a free-surface velocity of constant magnitude 0.5.

However, for a more general case where the vortex is in contact with a wall, our task is to determine its shape, and also to determine the constant value for the free-surface velocity. It is of course possible that 'contact' with the wall occurs at one or more isolated points only, as sketched for a plane wall in figure 3. This is not physically significant contact, and the vortex remains undistorted, as an effectively free object of a cylindrical shape. Our interest is rather in contact between vortex and wall which occurs over a segment of finite non-zero length, and distorts the shape of the vortex's free boundary.

In the present paper, we provide a numerical procedure which in principle enables solution for contact between a vortex and a wall of arbitrary shape. However, most of our calculations are performed for a wall consisting of a parabolic dish. If this parabola

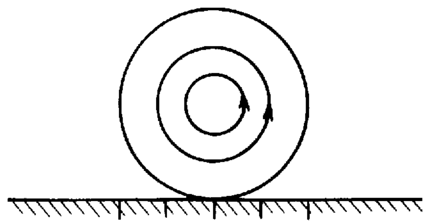

FIGURE 3. Free (circular) vortex in single-point contact with a plane. The vertical scale is the same as the horizontal scale.

is convex toward the vortex, the vortex can touch it at its tip without distortion, and the situation is essentially the same as in figure 3. This is also true if the parabola is concave toward the vortex, so long as its tip radius of curvature exceeds 1, i.e. exceeds the (non-dimensional) radius of a free vortex, since again such a free vortex can sit inside the dish, as in figure 4(*a*), with contact only at a single point. This includes the limiting case of a plane wall (dish of infinite radius of curvature) as in figure 3.

The problem only becomes interesting when the radius of curvature of the dish is less than 1, and we provide results for the shape and contact length of the vortex which lies inside such a dish. There is a one-parameter family of such solutions, the parameter being the non-dimensional dish radius of curvature, i.e. the ratio between the actual dish radius and the radius of the free vortex.

Results are also obtained for vortices in contact with a corner between two planes meeting at a general angle. Again, there is a one-parameter family, the parameter now being the corner angle. If that angle is 180°, i.e. the wall is plane, we retrieve the result that the only solution has point contact. As the corner angle decreases from 180°, the contact segment increases in length.

2. Mathematical formulation

The general task is to solve the problem as sketched in figure 2 for a vortex V of given vorticity and area, whose closed boundary curve $C = F + W$ consists of a free surface F and a contact segment with a wall W. We shall assume that the non-dimensionalization with respect to the actual vorticity ω and lengthscale R has already taken place. Then the problem is described in terms of a stream function $\psi(x, y)$ satisfying

$$\nabla^2\psi = -1 \tag{2.1}$$

inside C, with boundary conditions

$$\psi = 0 \tag{2.2}$$

on all of C, and

$$\frac{\partial\psi}{\partial n} = -U \tag{2.3}$$

on the free-surface portion F of C. Here U is the constant value of the free-surface velocity magnitude; note that (2.3) actually specifies that the tangential velocity is U, while (2.2) states that the normal velocity is zero. The constant U must be determined as part of the solution to the problem.

If all of C were known, the problem of solving the Poisson equation (2.1) subject to the Dirichlet boundary condition (2.2) would have a unique solution which is easy to find by various numerical tools. But the portion F is not known in advance, and our task is to vary the shape of F until the additional boundary condition (2.3) is satisfied on F. Not only is the actual shape of F unknown, but the location of its contact point with W is also unknown, as well as the value of the constant U.

The above problem can be reduced to that of solving an integral equation, in a number of ways. We have used a method similar to the one derived by Teles Da Silva & Peregrine (1988) to study waves on a shearing flow. The idea is to reduce the problem to one for Laplace's equation by subtraction of a particular solution. Thus if we write

$$\psi = \Psi - \tfrac{1}{2}y^2 \tag{2.4}$$

then $\nabla^2\Psi(x,y) = 0$. Now the quantity $w(z) = u - iv = \Psi_y + i\Psi_x$ is an analytic function of $x + iy$, where the fluid velocity vector is $(u-y, v)$. Hence by Cauchy's theorem, when z is on C,

$$w(z) = \frac{1}{\pi i}\oint_C \frac{w(\zeta)\,d\zeta}{\zeta - z} \tag{2.5}$$

with a Cauchy principal-value interpretation.

Suppose that the curve C is parametrized by $x = X(t)$, $y = Y(t)$ where t measures arclength, and ranges from $t = 0$ to $t = L$ say, where L is the total length of the perimeter of the wall and vortex. Then

$$X'(t)^2 + Y(t)^2 = 1. \tag{2.6}$$

We now consider u and v to be functions of t. Now the real part of (2.5) is

$$u(t) = \frac{1}{\pi}\int_0^L \frac{(X(s)-X(t))(u(s)\,Y'(s)-v(s)\,X'(s))-(Y(s)-Y(t))(u(s)\,X'(s)+v(s)\,Y'(s))}{(X(s)-X(t))^2+(Y(s)-Y(t))^2}\,ds, \tag{2.7}$$

an equation that holds for all t defining C, i.e. $0 < t < L$. The impermeability condition (2.2) can be written

$$v(t) = (u(t) - Y(t))\,Y'(t)/X'(t), \tag{2.8}$$

and this also holds everywhere on C.

The frec-surface condition of velocity magnitude U guarantees in the present case that the velocity vector is (UX', UY') on the free surface. Using this for the x-component, we have $u - Y = UX'$ on F. We now assume that the wall W is defined by a function $f(x)$ such that $y = f(x)$ on the wall, i.e. $Y(t) = f(X(t))$, say for $0 \leqslant t < \beta L$, where β is a parameter specifying what fraction of the total perimeter L of C is occupied by the wall. We can combine this wall-geometry specification on W with the free-surface condition on F as a 'formula' for $Y(t)$ valid everywhere on $C = W + F$, namely

$$Y(t) = \begin{cases} f(X(t)) & \text{if } \; 0 < t < \beta L; \\ u(t) - UX'(t) & \text{if } \; \beta L < t < L. \end{cases} \tag{2.9}$$

Equations (2.6)–(2.9) are a complete boundary-integral equivalent of the original boundary-value problem (2.1)–(2.3). Our interpretation of these equations as integral equations is as follows. Suppose that we consider the pair of functions $(u(t), X'(t))$ as our fundamental unknowns, to be determined over the whole boundary C, i.e. for $0 < t < L$. Then (2.9) yields $Y(t)$, and then (2.8) yields $v(t)$, directly in terms of these unknowns. Now we have two further equations, (2.6) and (2.7), which must both hold for all t, so providing a pair of nonlinear integral equations to determine $u(t)$ and $X'(t)$.

Of course there are also some new unknown constants that we have introduced into this problem, namely U, L and β, but these are determined as follows. The free-surface velocity U is a proper unknown, and is simply added to the list of unknown variables when the problem is discretized, as discussed below. The net perimeter length L cannot be specified separately from the requirement that the net area of the vortex take the value π. Hence we allow L to also be one of our discrete unknowns, but add a normalizing equation on the area, namely

$$\int_0^L Y(t)\, X'(t)\, \mathrm{d}t = \pi. \tag{2.10}$$

Finally, the parameter β measuring what fraction of the vortex's perimeter is in contact with the wall is bound up with the specification of the wall geometry, and we choose to determine that geometry (in part) inversely. This is most easily described for a special case. In the special case of a parabolic dish

$$f(x) = \frac{1}{2r} x^2, \tag{2.11}$$

where r is the dish radius of curvature, we allow r to be an unknown, and specify β instead. That is, we treat β as the only input parameter. Once the problem is solved, the radius r is available as an output quantity, as well as all quantities associated with the vortex shape. A similar procedure can be used for any single-parameter family of wall geometries. Furthermore, an n-parameter family of wall geometries can be reduced to a single-parameter family of wall geometries by fixing $n-1$ parameters.

3. Numerical procedure

We seek a numerical solution of the nonlinear integrodifferential system (2.6)–(2.9). First we define N distinct mesh points on the boundary C by specifying values of the arclength parameter $t = S_I$, where

$$S_I = L\frac{I}{N}, \quad I = 0, \ldots, N. \tag{3.1}$$

Note that $t = S_N = L$ and $t = S_0 = 0$ must correspond to the same point of space. We assume that $\beta = M/N = S_M/L$ for some M, so that there are exactly $M+1$ mesh points $t = S_0, S_1, S_2, \ldots, S_M$ on the wall and exactly $N-M+1$ mesh points $t = S_M, S_{M+1}, \ldots, S_N$ on the free surface. We shall also make use of the intermediate mesh points $S_{I-\frac{1}{2}} = \frac{1}{2}(S_{I-1}+S_I)$, $I = 1, 2, \ldots, N$.

We now define the $2N$ corresponding fundamental unknown quantities

$$u_I = u(S_I), \quad I = 1, 2, \ldots, N \tag{3.2}$$

and

$$X'_I = X'(S_I), \quad I = 1, 2, \ldots, N. \tag{3.3}$$

Let us suppose temporarily that these quantities are known.

We then estimate the value of the actual x-coordinate $X_I = X(S_I)$ by the trapezoidal rule, i.e. $X_0 = 0$ and

$$X_I = X_{I-1} + X'(S_{I-\frac{1}{2}})\, L/N, \quad I = 1, 2, \ldots, N, \tag{3.4}$$

where $X_{I-\frac{1}{2}}$ is evaluated from X_I by a four-point interpolation formula. We now use (2.6), (2.8) and (2.9) to estimate $Y_I = Y(S_I)$ and $v_I = v(S_I)$ in terms of u_I and $X'(I)$. Next we estimate $X'(S_{I-\frac{1}{2}})$, $Y(S_{I-\frac{1}{2}})$, $u(S_{I-\frac{1}{2}})$ by four-point interpolation.

Next we satisfy (2.7) at the point $t = S_{I-\frac{1}{2}}$, $I = 1, 2, \ldots, N$ by applying the trapezoidal rule to (2.7), with a sum over the points $s = S_J$, $J = 0, 1, 2, \ldots, N$. The symmetry of the discretization and of the trapezoidal rule with respect to the singularity of the integrand at $s = t$ enables us to evaluate this Cauchy principal value integral by ignoring the singularity, with an accuracy no less than a non-singular integral (Monacella 1967). This yields N nonlinear equations.

An extra N equations are obtained by expressing the fact that $Y'(t)$ is the derivative of $Y(t)$, using the centred difference formula

$$Y'(S_{I-\frac{1}{2}}) = (Y_I - Y_{I-1})\, L/N, \quad I = 1, 2, \ldots, N. \tag{3.5}$$

If all of the constants in the formulation were known, this would constitute $2N$ equations in $2N$ unknowns, so closing the system. However, U and L are also unknown, as is also one parameter within the specification of the wall function $f(x)$. Hence there are actually $2N+3$ unknowns. The extra three equations are as follows.

First we demand that the net area be π; that is, a trapezoidal approximation to (2.10) holds. Next we impose the continuity of Y' at both junction points between wall and free surface by a linear extrapolation formula.

The final effect is to yield $2N+3$ equations in $2N+3$ unknowns. In practice, when solving for problems with symmetry, it is possible by images to reduce this system to $N+3$ unknowns before solving by Newton's method.

4. Discussion of results

We first use the numerical scheme of §3 to compute solutions for the parabolic dish (2.11). Most calculations were performed with $N = 120$. We repeated the calculations for various values of N and checked that all the results presented are correct to at least two decimal places.

Typical solutions are shown in figure 4. For $r \geqslant 1$, the vortex touches the parabola at one point only, and is undistorted from the free circular shape. Figure 4(a) shows this solution at $r = 1$. For $r < 1$, the vortex is joined to the dish along a finite segment, and distorted. The length of this segment increases as r decreases, see figure 4.

In figures 5 and 6, we show values of the free-surface speed U and the wall perimeter fraction β versus r; note that these results are actually computed with β as an input parameter, and r output, as discussed in §3. For $r \geqslant 1$, $U = 0.5$, the free-vortex value, and as r decreases below 1, U decreases slowly. As r approaches zero, our scheme became more sensitive and accurate solutions could not be calculated for $r < 0.15$ with $N = 120$. However the results of figures 5 and 6 suggest that $U \to 0$ and $\beta \to 1$ as $r \to 0$.

Next we use the numerical method to compute solutions for vortices in contact with a corner between two planes. There is a solution for each value of the corner angle γ. Typical profiles are shown in figure 7. For $\gamma \geqslant \pi$, the solution is again a free undistorted cylindrical vortex. Figure 3 shows that solution for $\gamma = \pi$. As γ decreases from π, contact occurs along a segment of finite length, whose length increases as γ decreases, see figure 7. Figure 8 shows the wall perimeter fraction β versus γ. Our scheme became more sensitive for acute angles γ. However the results suggest that $\beta \to 1$ as $\gamma \to 0$.

Finally we present a simple check on the validity of our numerical procedure by computing the flow for a circular dish

$$f(x) = r - (r^2 - x^2)^{\frac{1}{2}}. \tag{4.1}$$

The exact solution in this case is a circle of unit radius. The input parameter β simply

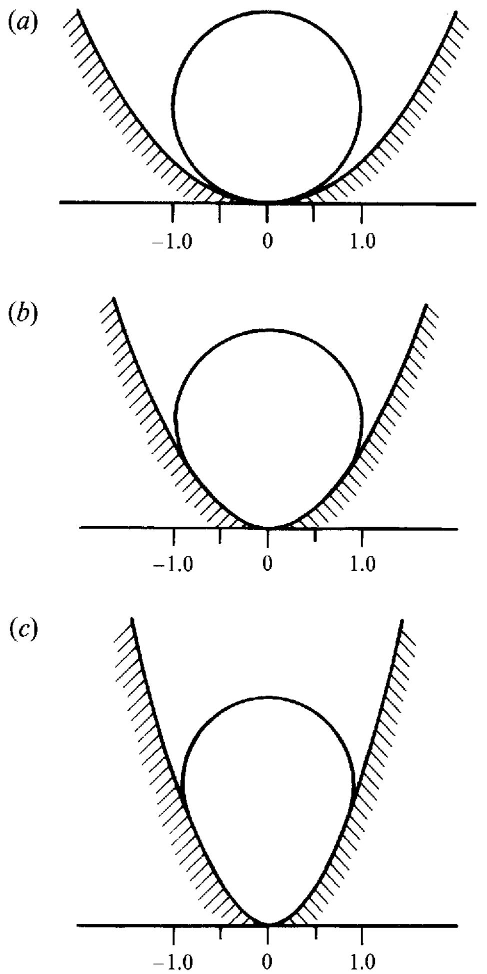

FIGURE 4. Vortices in contact with parabolas of various radii of curvature. The vertical scale is the same as the horizontal scale. (*a*) $r = 1.0$, (*b*) $r = 0.58$, (*c*) $r = 0.25$.

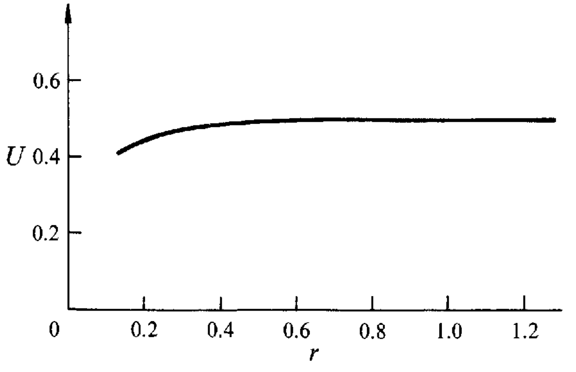

FIGURE 5. Free-surface velocity magnitude U as a function of the parabola's (relative) radius of curvature r.

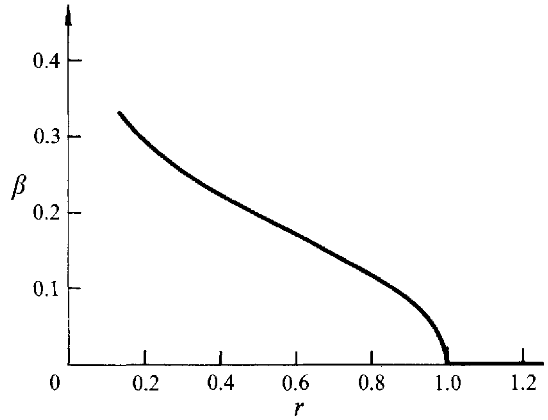

FIGURE 6. Fraction β of vortex perimeter in contact with parabolic wall, as a function of wall radius of curvature r.

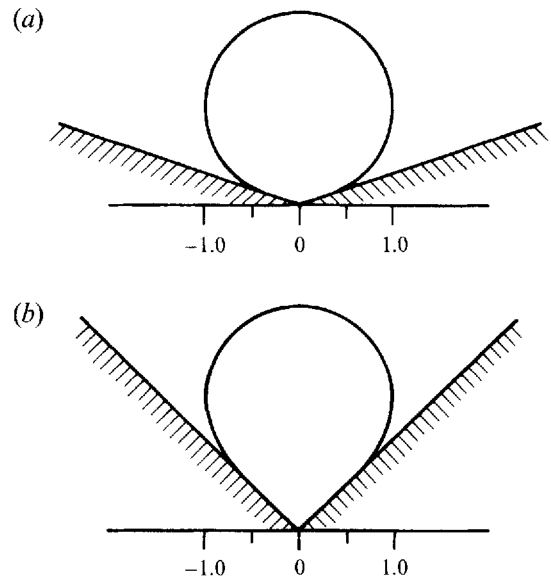

FIGURE 7. Vortices in contact with corners of various angles. The vertical scale is the same as the horizontal scale. (*a*) $\gamma = 2.49$, (*b*) $\gamma = 1.59$.

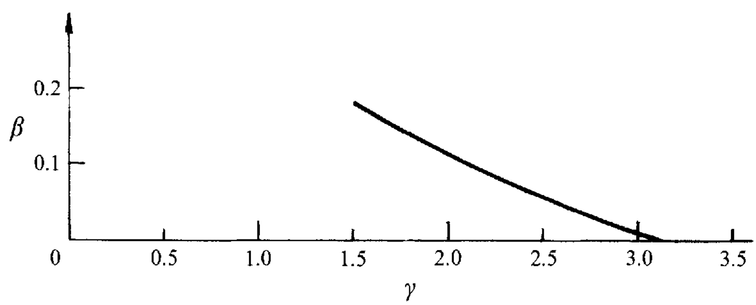

FIGURE 8. Fraction β of vortex perimeter in contact with the corner, as a function of the angle γ.

determines what fraction of that circle is identified with the dish (4.1), but the output parameter should always take the value $r = 1$. Our numerical results for $N = 120$ give $r = 0.998$ for $\beta = 0.04$ and $r = 1.000$ for $\beta = 0.16$.

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